

## Theory of Biaxial Graded-Index Optical Fiber

Stephen F. Kawalko, Student Member, IEEE and  
Piergiorgio L.E. Uslenghi, Fellow, IEEE

P-27

### Abstract

The problem of wave propagation in a biaxial graded-index fiber with circular symmetry is considered. The problem is formulated in terms of four first-order differential equations for the tangential components of the electric and magnetic fields. A general solution method for solving systems of differential equations is presented. This solution method is then used to solve the system of equations for a particular example of a biaxial graded-index fiber. Numerical results for the propagation constant in the fiber are also given.

### I. Introduction

The optical fiber has become a much studied transmission system due to its property of wave guidance with low loss. In recent years it has been shown that introducing anisotropies into the dielectric medium of the fiber produces several interesting features, such as control of power flow and reduction of peak attenuation near cutoff.

Typically the analysis of wave propagation in a cylindrical dielectric waveguide such as an optical fiber is performed using a wave equation formulation. For the simple case of a step-index fiber a detailed analysis, including dispersion relations, cutoff conditions and mode designations, is presented by Snitzer [1]. Paul and Shevgaonkar [2] present a similar analysis for a uniaxial step-index fiber and also perform a perturbation analysis to

---

The authors are with the Department of Electrical Engineering and Computer Science, University of Illinois at Chicago, Chicago, IL 60680

This research was supported by the Pacific Missile Test Center and the NASA-Ames Research Center under grant NAG2-544

determine the modal attenuation constants. These are the only two cases for which exact solutions are known.

For inhomogeneous fibers no exact solutions are known. For the case of an isotropic graded-index fiber several approximate analytic solution methods are available. These approximate solutions all share the common assumption that the fiber is infinite in extent. In addition if the permittivity is assumed to vary slowly over the distance of one wavelength the wave equation formulation simplifies to an associated scalar wave equation. If the permittivity profile is parabolic the solution to the scalar wave equation can be written in terms of either Laguerre polynomials [3] if cylindrical coordinates are used or Hermite polynomials [4] if rectangular coordinates are used. For arbitrary permittivity profiles the scalar wave equation can be solved using the well known WKB solution method [5], [6]. For a parabolic permittivity profile all three solution methods give identical results. Under the assumption that the fields are far from cutoff Kurtz and Streifer [7], [8] have shown that a solution to the full vector problem can be written in terms of either Laguerre polynomials if the permittivity profile is quadratic or asymptotically in terms of Bessel and Airy functions for arbitrary permittivity profiles which decrease slowly and monotonically. A comparison of the vector and scalar solutions for the quadratic permittivity profile implies the vector modes can be obtained by simply renumbering the scalar modes [9]. Using the renumbered scalar modes as a basis Hashimoto [10] and Ikuno [11] have developed two slightly different iterative methods which can be used to solve the full vector problem for an isotropic graded-index fiber.

An alternate formulation of the problem is to write the four first-order differential

equations for the tangential field components as a first-order matrix differential equation. For a step-index fiber with uniaxial core and cladding Tønning [12] has shown that the matrix formulation can be solved exactly in terms of Bessel functions. For isotropic graded-index fibers with arbitrary permittivity profiles Yeh and Lingren [13] have indirectly used the matrix formulation in developing a numerical solution method based on the concept of stratification. Using the concept of transition matrices Tønning [14] has developed a numerical procedure which can be used to solve the matrix differential equation for isotropic graded-index fibers.

## II. Formulation of the Problem

Consider a circularly symmetric optical fiber with the geometry shown in Figure 1. In the core,  $0 \leq \rho \leq a$ , the permittivity is given by  $\epsilon_0 \bar{\epsilon}_r(\rho)$  where  $\epsilon_0$  is the permittivity of free space and  $\bar{\epsilon}_r(\rho)$  is the relative permittivity tensor of the core and is a function of  $\rho$  only. In the cladding,  $a \leq \rho \leq b$ , the permittivity is given by  $\epsilon_0 \epsilon_c$  where  $\epsilon_c$  is the relative permittivity of the cladding and is assumed to be constant. In both the core and the cladding the permeability is  $\mu_0$ , the permeability of free space. For convenience, the external radius of the cladding,  $b$ , is assumed to be sufficiently large in comparison to the radius of the core,  $a$ , so that it is not necessary to impose boundary conditions at the air-cladding boundary.

Consider the case where the relative permittivity tensor in the core is given by

$$\bar{\epsilon}_r(\rho) = \begin{pmatrix} \epsilon_1(\rho) & 0 & 0 \\ 0 & \epsilon_2(\rho) & 0 \\ 0 & 0 & \epsilon_3(\rho) \end{pmatrix}_{\rho, \phi, z} ; \quad (1)$$

where  $\epsilon_1(\rho)$ ,  $\epsilon_2(\rho)$  and  $\epsilon_3(\rho)$  are the relative permittivities in the  $\rho$ ,  $\phi$  and  $z$  directions respectively. In general the relative permittivities are arbitrary functions of  $\rho$ . However, the choice of cylindrical coordinates requires  $\epsilon_1(\rho)$  and  $\epsilon_2(\rho)$  be equal at  $\rho = 0$ .

For time harmonic fields in a source free region, Maxwell's equations can be written as

$$\nabla \times \mathbf{H} = j\omega\epsilon_0\bar{\epsilon}_r\mathbf{E}, \quad (2a)$$

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}, \quad (2b)$$

where  $\omega$  is the angular frequency. If the  $z$  and  $\phi$  dependence of the fields is given by

$$e^{-j\beta z + jm\phi},$$

where  $\beta$  is the longitudinal wavenumber and  $m$  is any integer, then for cylindrical coordinates Maxwell's equations can be written in component form as

$$\frac{m}{\rho}H_z + \beta H_\phi = \omega\epsilon_0\epsilon_1 E_\rho, \quad (3a)$$

$$-j\beta H_\rho - \frac{dH_z}{d\rho} = j\omega\epsilon_0\epsilon_2 E_\phi, \quad (3b)$$

$$\frac{1}{\rho} \frac{d}{d\rho}(\rho H_\phi) - \frac{j m}{\rho} H_\rho = j\omega\epsilon_0\epsilon_3 E_z, \quad (3c)$$

$$\frac{m}{\rho}E_z + \beta E_\phi = -\omega\mu_0 H_\rho, \quad (3d)$$

$$j\beta E_\rho + \frac{dE_z}{d\rho} = j\omega\mu_0 H_\phi, \quad (3e)$$

$$\frac{1}{\rho} \frac{d}{d\rho}(\rho E_\phi) - \frac{j m}{\rho} E_\rho = -j\omega\mu_0 H_z. \quad (3f)$$

The remainder of the problem can now be formulated in two different ways. If the transverse field components  $E_\rho$ ,  $E_\phi$ ,  $H_\rho$  and  $H_\phi$  are eliminated from eqs. (3) we obtain a pair of coupled second-order differential equations for the longitudinal field components  $E_z$  and  $H_z$ . Alternately, if the radial components  $E_\rho$  and  $H_\rho$  are eliminated we obtain a system of four first-order differential equations for the tangential field components  $E_z$ ,  $E_\phi$ ,  $H_z$  and  $H_\phi$ .

First consider the coupled wave equation formulation. It is convenient to define a normalized magnetic field  $h = Z_0 H$  where  $Z_0 = \sqrt{\mu_0/\epsilon_0}$  is the impedance of free space.

Solving eqs. (3a), (3b), (3d) and (3e) for  $E_\rho$ ,  $E_\phi$ ,  $h_\rho$  and  $h_\phi$  gives

$$E_\rho = \frac{1}{k_{t1}^2} \left[ \frac{mk_0}{\rho} h_z - j\beta \frac{dE_z}{d\rho} \right], \quad (4a)$$

$$h_\phi = \frac{1}{k_{t1}^2} \left[ \frac{m\beta}{\rho} h_z - jk_0\epsilon_1 \frac{dE_z}{d\rho} \right], \quad (4b)$$

$$E_\phi = \frac{1}{k_{t2}^2} \left[ \frac{m\beta}{\rho} E_z + jk_0 \frac{dh_z}{d\rho} \right], \quad (4c)$$

$$h_\rho = \frac{1}{k_{t2}^2} \left[ \frac{-mk_0\epsilon_2}{\rho} E_z - j\beta \frac{dh_z}{d\rho} \right] \quad (4d)$$

where  $k_0 = \omega\sqrt{\epsilon_0\mu_0}$  is the free space wavenumber and  $k_{tn}^2 = k_0^2\epsilon_n(\rho) - \beta^2$ ,  $n = 1, 2$  is the transverse wave number. Substituting the expressions for  $E_\rho$ ,  $E_\phi$ ,  $h_\rho$  and  $h_\phi$  given by eqs. (4) into eqs. (2c) and (2f) and making a change of variable from  $\rho$  to a normalized radius  $r = \rho/a$  results in the following pair of coupled differential equations for  $E_z$  and  $h_z$

$$E_z'' + f_1(r)E_z' + \Lambda^2 g_1(r)E_z = p_2(r)h_z' + q_2(r)h_z, \quad (5a)$$

$$h_z'' + f_2(r)h_z' + \Lambda^2 g_2(r)h_z = p_1(r)E_z' + q_1(r)E_z, \quad (5b)$$

where  $' = d/dr$ ,  $\Lambda^2 = (k_0 a)^2$ ,  $\kappa = \beta/k_0$  and

$$f_1(r) = \frac{1}{r} - \frac{\kappa^2 \epsilon_1'(r)}{\epsilon_1(r)[\epsilon_1(r) - \kappa^2]}, \quad (6a)$$

$$f_2(r) = \frac{1}{r} - \frac{\epsilon_2'(r)}{\epsilon_2(r) - \kappa^2}, \quad (6b)$$

$$g_1(r) = \frac{\epsilon_3(r)}{\epsilon_1(r)} [\epsilon_1(r) - \kappa^2] \left[ 1 - \frac{m^2 \epsilon_2(r)}{\Lambda^2 \epsilon_3(r) [\epsilon_2(r) - \kappa^2] r^2} \right], \quad (6c)$$

$$g_2(r) = [\epsilon_2(r) - \kappa^2] \left[ 1 - \frac{m^2}{\Lambda^2 [\epsilon_1(r) - \kappa^2] r^2} \right], \quad (6d)$$

$$p_1(r) = \frac{j m \kappa}{r} \left[ \frac{\epsilon_1(r) - \epsilon_2(r)}{\epsilon_1(r) - \kappa^2} \right], \quad (6e)$$

$$p_2(r) = -\frac{j m \kappa}{\epsilon_1(r) r} \left[ \frac{\epsilon_2(r) - \epsilon_1(r)}{\epsilon_2(r) - \kappa^2} \right], \quad (6f)$$

$$q_1(r) = -\frac{j m \kappa}{r} \left[ \frac{\epsilon_2'(r)}{\epsilon_2(r) - \kappa^2} \right], \quad (6g)$$

$$q_2(r) = \frac{j m \kappa}{\epsilon_1(r) r} \left[ \frac{\epsilon_1'(r)}{\epsilon_1(r) - \kappa^2} \right]. \quad (6h)$$

The equations for  $E_z$  and  $h_z$  become uncoupled for three particular cases. For the so-called meridional modes  $m$  is equal to zero and therefore from eqs. (6e,f,g,h) so are the functions  $p_1(r)$ ,  $p_2(r)$ ,  $q_1(r)$  and  $q_2(r)$ . For isotropic and uniaxial step-index fibers  $\epsilon_1$  and  $\epsilon_2$  are equal and constant and again from eqs. (6e,f,g,h) the functions  $p_1(r)$ ,  $p_2(r)$ ,  $q_1(r)$  and  $q_2(r)$  are zero.

In general a solution of eqs. (5) for arbitrary permittivity profiles is not possible. It is possible to obtain a fourth-order differential equation for either  $E_z$  or  $h_z$  by eliminating  $h_z$  or  $E_z$  from eqs. (5). However, the complexity of the resulting equation precludes the determination of a solution. For meridional modes a direct series solution of the uncoupled equations is possible. However, due to the poles in the functions  $f_1(r)$  and  $f_2(r)$  the resulting series solution will not be convergent for the entire core region. An exact solution of eqs. (5) is possible only for the case of a step-index fiber. For either an isotropic or uniaxial step-index fiber the coupled equations simplify to Bessel's differential equation.

In order to find an analytic solution of eqs. (5) some assumptions must be made. First, the cladding is neglected and the core is assumed to extend to infinity. This eliminates the need to impose boundary conditions on the solution at the core-cladding boundary. Second, the permittivities are assumed to be slowly varying functions of  $r$  over a distance of several wavelengths. This is equivalent to assuming  $\epsilon_i'(r) \approx 0$ . For the case of either an isotropic or a uniaxial graded-index fiber,  $\epsilon_1(r) = \epsilon_2(r)$ , application of the second assumption to eqs. (5)

results in the following equations for  $E_z$  and  $h_z$

$$E_z'' + \frac{1}{r}E_z' + \Lambda^2 g_1(r)E_z = 0, \quad (7a)$$

$$h_z'' + \frac{1}{r}h_z' + \Lambda^2 g_2(r)h_z = 0, \quad (7b)$$

where  $g_1(r)$  and  $g_2(r)$  are given by

$$g_1(r) = \frac{\epsilon_3(r)}{\epsilon_1(r)} \left[ \epsilon_1(r) - \kappa^2 \right] - \frac{m^2}{\Lambda^2 r^2}, \quad (8a)$$

$$g_2(r) = \epsilon_1(r) - \kappa^2 - \frac{m^2}{\Lambda^2 r^2}. \quad (8b)$$

For the case of a biaxial graded-index fiber,  $\epsilon_1(r) \neq \epsilon_2(r)$ , the previous assumption does not cause eqs. (5) to uncouple since  $p_1(r)$  and  $p_2(r)$  are not identically equal to zero.

Eqs. (7) can be solved easily using the well known Wentzel-Kramers-Brillouin (WKB) solution method [5], [6], [15]. The solutions obtained using the WKB method are not solutions of the full vector problem given by eqs. (5) but rather they are solutions of a related scalar problem given by eqs. (7). However, the vector solutions can be obtained by renumbering the solutions to the scalar problem [9].

For the case of a biaxial graded-index fiber the WKB solution method can be applied blindly to eqs. (5) and at most only two terms in the WKB expansion can be determined. Nevertheless, the term representing the phase of the WKB solution is not a well behaved function and therefore it is not reasonable to assume the WKB solution method remains valid under this condition. In order to solve Maxwell's equations for the case of a biaxial graded-index fiber an alternate formulation must be used.

Instead of eliminating the transverse field components from eqs. (3), eliminate the radial field components  $E_\rho$  and  $H_\rho$  and write the remaining four equations as a system of four

first-order differential equations in terms of the tangential components [14]. From the two algebraic equations, eqs. (3a) and (3b), the radial components can be written in terms of the tangential components as

$$E_\rho = \frac{1}{\omega\epsilon_0\epsilon_1} \left[ \frac{m}{\rho} H_z + \beta H_\phi \right], \quad (9a)$$

$$H_\rho = -\frac{1}{\omega\mu_0} \left[ \frac{m}{\rho} E_z + \beta H_\phi \right]. \quad (9b)$$

Using eqs. (9), the four remaining equations can be written as

$$\frac{dE_z}{ds} = -j \frac{m\kappa}{s\epsilon_1} h_z + \frac{j}{s\epsilon_1} (\epsilon_1 - \kappa^2) (sh_\phi), \quad (10a)$$

$$\frac{d}{ds}(sE_\phi) = \frac{j}{s\epsilon_1} (m^2 - \epsilon_1 s^2) h_z + j \frac{m\kappa}{s\epsilon_1} (sh_\phi), \quad (10b)$$

$$\frac{dh_z}{ds} = j \frac{m\kappa}{s} E_z - \frac{j}{s} (\epsilon_2 - \kappa^2) (sE_\phi), \quad (10c)$$

$$\frac{d}{ds}(sh_\phi) = -\frac{j}{s} (m^2 - \epsilon_3 s^2) E_z - j \frac{m\kappa}{s} (sE_\phi) \quad (10d)$$

where a change of variable from  $\rho$  to a normalized radius  $s = k_0\rho$  has been made. Eqs. (10)

can be written in matrix form as

$$\frac{d\mathbf{u}}{ds} = \frac{1}{s} \mathbf{A}(s) \mathbf{u}, \quad (11a)$$

where

$$\mathbf{u} = (E_z \quad sE_\phi \quad h_z \quad sh_\phi)^T \quad (11b)$$

and

$$\mathbf{A}(s) = \begin{pmatrix} 0 & 0 & -j \frac{m\kappa}{\epsilon_1} & \frac{j}{\epsilon_1} (\epsilon_1 - \kappa^2) \\ 0 & 0 & \frac{j}{\epsilon_1} (m^2 - \epsilon_1 s^2) & j \frac{m\kappa}{\epsilon_1} \\ j m \kappa & -j (\epsilon_2 - \kappa^2) & 0 & 0 \\ -j (m^2 - \epsilon_3 s^2) & -j m \kappa & 0 & 0 \end{pmatrix}. \quad (11c)$$

For the special case of meridional modes,  $m = 0$ , eqs. (10) can be separated into two systems each containing two equations. The first set corresponding to transverse magnetic



modes (TM) can be written in matrix form as

$$\frac{d\mathbf{u}^{(\text{TM})}}{ds} = \frac{1}{s} \mathbf{A}^{(\text{TM})}(s) \mathbf{u}^{(\text{TM})} \quad (12a)$$

where

$$\mathbf{u}^{(\text{TM})} = (E_z \quad sh_\phi)^T \quad (12b)$$

and

$$\mathbf{A}^{(\text{TM})}(s) = \begin{pmatrix} 0 & \frac{j}{\epsilon_1}(\epsilon_1 - \kappa^2) \\ j\epsilon_3 s^2 & 0 \end{pmatrix}. \quad (12c)$$

The second set corresponding to transverse electric modes (TE) can be written as

$$\frac{d\mathbf{u}^{(\text{TE})}}{ds} = \frac{1}{s} \mathbf{A}^{(\text{TE})}(s) \mathbf{u}^{(\text{TE})} \quad (13a)$$

where

$$\mathbf{u}^{(\text{TE})} = (h_z \quad sE_\phi)^T \quad (13b)$$

and

$$\mathbf{A}^{(\text{TE})}(s) = \begin{pmatrix} 0 & -j(\epsilon_2 - \kappa^2) \\ -js^2 & 0 \end{pmatrix}. \quad (13c)$$

The only known exact solutions of the matrix equation are for the cases of an isotropic and a uniaxial step-index fiber [12], [14]. These solutions are identical to the exact solutions of the wave equation formulation.

It is not readily apparent that the matrix equation is easier to solve than the wave equation formulation. As was mentioned earlier, a series solution for the wave equation formulation is possible only when the equations are uncoupled. However, for the meridional modes of a graded-index fiber no series solution will be convergent for the entire core region. In contrast, the system matrix  $\mathbf{A}(s)$  does not have any poles in the core region and therefore the series solutions will be convergent in the entire core region.

While the form of  $\mathbf{A}(x)$  guarantees a convergent series solution the series may not converge rapidly enough to use it in numerical computations. An alternate solution method is asymptotic partitioning of systems of equations [16]. This method involves the transformation of a system of linear first-order differential equations into a system of equations whose solutions are easier to find. The form of the solution method presented in the next section is based on the expansion of the a general system matrix  $\mathbf{A}(x)$  in terms of positive powers of  $x$ , in contrast to the usual form where the expansion is in terms of powers of  $1/x$  (see e.g. [16])

### III. Matrix Partitioning

Consider the following system of  $N$  linear differential equations

$$\frac{d\mathbf{u}}{dx} = \frac{1}{x^q} \mathbf{A}(x) \mathbf{u}(x) \quad \text{as } x \rightarrow 0 \quad (14)$$

where  $\mathbf{u}$  is a column vector,  $q$  is an integer greater than or equal to 1 and  $\mathbf{A}(x)$  is a  $N \times N$  matrix given by

$$\mathbf{A}(x) = \sum_{n=0}^{\infty} \mathbf{A}_n x^n \quad \text{as } x \rightarrow 0. \quad (15)$$

It is possible to simplify this system of equations by transforming them into some special differential equations whose solutions are easier to find. Let

$$\mathbf{u}(x) = \mathbf{P}(x) \mathbf{v}(x) \quad (16)$$

where  $\mathbf{v}$  is a column vector and  $\mathbf{P}(x)$  is a  $N \times N$  nonsingular matrix. Using eq. (16), the original problem given by eq. (14) can be transformed into

$$\frac{d\mathbf{v}}{dx} = \frac{1}{x^q} \mathbf{B}(x) \mathbf{v}(x) \quad (17)$$

where

$$\mathbf{B}(x) = \mathbf{P}(x)^{-1} \left[ \mathbf{A}(x)\mathbf{P}(x) - x^q \frac{d\mathbf{P}(x)}{dx} \right] \quad (18)$$

or more conveniently

$$x^q \frac{d\mathbf{P}(x)}{dx} = \mathbf{A}(x)\mathbf{P}(x) - \mathbf{P}(x)\mathbf{B}(x). \quad (19)$$

The matrix  $\mathbf{P}(x)$  is chosen so that  $\mathbf{B}(x)$  has a convenient form, either diagonal or Jordan canonical form. If  $\mathbf{B}(x)$  has either of these forms the solution of the transformed system for  $\mathbf{v}$  is trivially obtained. For example, if  $\mathbf{A}(x)$  is a constant matrix then  $\mathbf{P}(x)$  is also a constant matrix and eq. (18) is simply a similarity transformation. This implies  $\mathbf{P}(x)$  is chosen so that  $\mathbf{B}(x)$  is either the diagonal or the Jordan canonical form of  $\mathbf{A}(x)$ . In general, when  $\mathbf{A}(x)$  is not a constant matrix  $\mathbf{P}(x)$  is not a constant matrix and it is not clear from either eq. (18) or (19) how  $\mathbf{P}(x)$  should be chosen so  $\mathbf{B}(x)$  has the desired form.

In order to develop a procedure to find  $\mathbf{B}(x)$  and  $\mathbf{P}(x)$  start by expanding them as the following Taylor series

$$\begin{aligned} \mathbf{B}(x) &= \sum_{n=0}^{\infty} \mathbf{B}_n x^n \quad \text{as } x \rightarrow 0, \\ \mathbf{P}(x) &= \sum_{n=0}^{\infty} \mathbf{P}_n x^n \quad \text{as } x \rightarrow 0, \end{aligned} \quad (20)$$

where in general  $\mathbf{B}_0$  is a Jordan canonical matrix and  $\mathbf{B}_n$  is a diagonal matrix. Substituting eqs. (20) into eq. (19) and equating like powers of  $x$  gives

$$\mathbf{A}_0 \mathbf{P}_0 - \mathbf{P}_0 \mathbf{B}_0 = 0 \quad (21)$$

for  $x^0$  and

$$(n - q + 1) \mathbf{P}_{n-q+1} = \sum_{l=0}^n (\mathbf{A}_l \mathbf{P}_{n-l} - \mathbf{P}_l \mathbf{B}_{n-l}) \quad (22)$$

for  $x^n$ ,  $n \geq 1$ , where  $\mathbf{P}_{n-q+1} = 0$  for  $n - q + 1 < 0$ . Eqs. (21) and (22) define an iterative procedure to find the coefficient matrices for the series expansions of  $\mathbf{B}(x)$  and  $\mathbf{P}(x)$  so that either eq. (18) or (19) is satisfied. Eq. (21) can be rewritten as

$$\mathbf{B}_0 = \mathbf{P}_0^{-1} \mathbf{A}_0 \mathbf{P}_0 \quad (23)$$

which implies  $\mathbf{P}_0$  is chosen so that  $\mathbf{B}_0$  is either the diagonal or Jordan canonical form of  $\mathbf{A}_0$ . With some algebraic manipulations eq. (22) can be written more conveniently as

$$\mathbf{B}_0 \mathbf{W}_n - \mathbf{W}_n \mathbf{B}_0 = (n - q + 1) \mathbf{W}_{n-q+1} + \mathbf{B}_n - \mathbf{F}_n \quad (24)$$

where the matrices  $\mathbf{W}_n$  and  $\mathbf{F}_n$  are defined as

$$\mathbf{W}_n = \mathbf{P}_0^{-1} \mathbf{P}_n \quad (25)$$

and

$$\mathbf{F}_n = \mathbf{P}_0^{-1} \mathbf{A}_n \mathbf{P}_0 + \mathbf{P}_0^{-1} \sum_{l=1}^{n-1} (\mathbf{A}_{n-l} \mathbf{P}_l - \mathbf{P}_l \mathbf{B}_{n-l}). \quad (26)$$

Notice that the unknowns in eq. (24) are the matrices  $\mathbf{B}_n$  and  $\mathbf{W}_n$  and that the matrices  $\mathbf{W}_{n-q+1}$  and  $\mathbf{F}_n$  depend solely on matrices found in previous iterations. Since by definition  $\mathbf{B}_n$  is a diagonal matrix eq. (24) can be solved easily for  $\mathbf{B}_n$  and  $\mathbf{W}_n$  by setting the diagonal elements of  $\mathbf{B}_n$  and  $\mathbf{F}_n$  equal to each other and then solving for  $\mathbf{W}_n$  from what remains of eq. (24). Since the form of  $\mathbf{B}_0$  is known in advance an explicit solution in terms of  $\mathbf{B}_0$  and  $\mathbf{F}_n$  can be found for  $\mathbf{W}_n$ .

Consider the special case where  $\mathbf{B}_0$  is a diagonal matrix and  $q = 1$ . This corresponds to the form of the matrix differential equation (11) which we want to solve. While it is not obvious from eq. (11c) that  $\mathbf{B}_0$  will be a diagonal matrix, it will be shown later that this is indeed the case.

When  $\mathbf{B}_0$  is a diagonal matrix the expression  $\mathbf{B}_0\mathbf{W}_n - \mathbf{W}_n\mathbf{B}_0$  has zeroes along it's main diagonal and does not depend upon the elements along the main diagonal of  $\mathbf{W}_n$ .

The solution to eq. (24) can be easily written as

$$(\mathbf{B}_n)_{ij} = \begin{cases} (\mathbf{F}_n)_{ii}, & i = j; \\ 0, & i \neq j, \end{cases} \quad (27)$$

and

$$(\mathbf{W}_n)_{ij} = \begin{cases} 0 & i = j; \\ -\frac{1}{\lambda_i - \lambda_j - n} (\mathbf{F}_n)_{ij} & i \neq j, \end{cases} \quad (28)$$

where  $\lambda_i$ ,  $i = 1, 2, \dots, N$  are the eigenvalues of  $\mathbf{A}_0$ . One potential problem exists with this solution. If  $\lambda_i - \lambda_j - n = 0$  and  $(\mathbf{F}_n)_{ij} \neq 0$  for some particular values of  $i$ ,  $j$  and  $n$  then it may not be possible to find  $\mathbf{W}_n$  and therefore a solution may not be possible.

Consider a biaxial graded-index fiber with permittivity profiles of the power law type given by

$$\epsilon_i(r) = \epsilon_i(1 - 2\Delta_i r^{\alpha_i}) \quad i = 1, 2, 3 \quad (29)$$

where  $\epsilon_i = \epsilon_i(0)$  and  $\Delta_i = (\epsilon_i - \epsilon_c)/2\epsilon_i$ . Since the choice of the coordinate system requires  $\epsilon_1(0) = \epsilon_2(0)$ , the definition of  $\Delta_i$  requires that  $\Delta_1$  and  $\Delta_2$  be equal. Then, for this choice of permittivity profiles  $\epsilon_1(r)$  and  $\epsilon_2(r)$  are not equal only when  $\alpha_1 \neq \alpha_2$ . The case of a step-index fiber exists as a special case to the power law profiles in the limit as  $\alpha_i \rightarrow \infty$ , or equivalently by setting  $\Delta_i = 0$ .

Now let us solve the matrix equation for the transverse modes in a biaxial fiber where the permittivity profiles are parabolic. The relative permittivity profiles can be written in terms of the normalized radius  $s = k_0\rho = (k_0a)r$  as

$$\epsilon_i(s) = \epsilon_i(1 - 2\Delta_i^0 s^2) \quad i = 1, 2, 3 \quad (30)$$

where  $\Delta_i^0 = \Delta_i/(k_0 a)^2$ . Strictly speaking this choice for  $\epsilon_i(s)$  does not produce a biaxial fiber since by definition  $\epsilon_1 = \epsilon_2$  and  $\Delta_1^0 = \Delta_2^0$ . Since  $\epsilon_1(s)$  and  $\epsilon_2(s)$  do not appear together in the matrix equations for the transverse modes, eqs. (12) and (13), it is not necessary to set  $\epsilon_2$  equal to  $\epsilon_1$ . However, in the final result it is necessary to replace  $\epsilon_2$  by  $\epsilon_1$  and either set  $\Delta_1^0 = 0$  and obtain the solution for a biaxial fiber where  $\epsilon_1(s)$  is constant and  $\epsilon_2(s)$  is parabolic or set  $\Delta_2^0 = 0$  and obtain the solution for the case where  $\epsilon_1(s)$  is parabolic and  $\epsilon_2(s)$  is constant. The first term in the series expansions for  $\mathbf{A}^{(\text{TM})}(s)$  and  $\mathbf{A}^{(\text{TE})}(s)$  is given by

$$\mathbf{A}_0^i = \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} \quad (31a)$$

where

$$\tau = \begin{cases} \frac{j}{\epsilon_1}(\epsilon_1 - \kappa^2), & i = \text{TM}; \\ -j(\epsilon_2 - \kappa^2), & i = \text{TE}. \end{cases} \quad (31b)$$

Since the two eigenvalues of  $\mathbf{A}_0$  are both equal to zero it is not possible to find a non-zero  $\mathbf{P}_0$  such that  $\mathbf{B}_0$  is a diagonal matrix. Instead  $\mathbf{P}_0$  must be chosen so that  $\mathbf{B}_0$  is a Jordan canonical matrix. For  $\mathbf{A}_0$  as given by eq. (31) choose  $\mathbf{P}_0$  as

$$\mathbf{P}_0 = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix} \quad (32)$$

so that

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (33)$$

is a Jordan canonical matrix. After four iterations, the solution for the TM case is found to be:

$$E_z = \frac{j}{\epsilon_3} k_{N1}^2 C_1 \left\{ 1 - \frac{\epsilon_3}{\epsilon_1} \frac{k_{N1}^2}{4} s^2 + \left[ \left( \frac{\epsilon_3}{\epsilon_1} \right)^2 \frac{k_{N1}^4}{64} + \frac{\epsilon_3}{\epsilon_1} \Delta_3^0 \frac{k_{N1}^2}{8} \right] s^4 \right\} e^{\frac{\epsilon_3}{\epsilon_1} (\Delta_1^0 \kappa^2) \frac{s^4}{4}}, \quad (34a)$$

$$s h_\phi = -C_1 \left\{ \frac{\epsilon_3}{\epsilon_1} \frac{k_{N1}^2}{2} s^2 - \left[ \left( \frac{\epsilon_3}{\epsilon_1} \right)^2 \frac{k_{N1}^4}{16} + \frac{\epsilon_3}{\epsilon_1} \Delta_3^0 \frac{k_{N1}^2}{2} \right] s^4 \right\} e^{\frac{\epsilon_3}{\epsilon_1} (\Delta_1^0 \kappa^2) \frac{s^4}{4}}, \quad (34b)$$

where  $k_{N1}^2 = \epsilon_1 - \kappa^2$  and  $C_1$  is a constant. The solution for the TE case is

$$h_z = -j k_{N2}^2 C_1 \left[ 1 - \frac{k_{N2}^2}{4} s^2 + \frac{k_{N2}^4}{64} s^4 \right] e^{\epsilon_2 \Delta_2^0 \frac{s^4}{4}}, \quad (35a)$$

$$s E_\phi = -C_1 \left[ \frac{k_{N2}^2}{2} s^2 - \frac{k_{N2}^4}{16} s^4 \right] e^{\epsilon_2 \Delta_2^0 \frac{s^4}{4}}, \quad (35b)$$

where  $k_{N2}^2 = \epsilon_2 - \kappa^2$ .

Now consider the solution of the matrix equation for hybrid modes. For all permittivity profiles the first term in the series expansion of  $\mathbf{A}(s)$  is

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 0 & -j \frac{m\kappa}{\epsilon_1} & j \frac{k_{N1}^2}{\epsilon_1} \\ 0 & 0 & \frac{j m^2}{\epsilon_1} & j \frac{m\kappa}{\epsilon_1} \\ j m\kappa & -j k_{N1}^2 & 0 & 0 \\ -j m^2 & -j m\kappa & 0 & 0 \end{pmatrix} \quad (36)$$

where  $k_{N1}^2 = \epsilon_1 - \kappa^2$  and the eigenvalues of  $\mathbf{A}_0$  are  $\pm m$ ,  $m \neq 0$ . Since the eigenvalues are repeated, in general, the choice for  $\mathbf{P}_0$  should at best cause  $\mathbf{B}_0$  to be a Jordan canonical matrix. This is the only restriction placed on the form of  $\mathbf{P}_0$  by the solution method. Any  $\mathbf{P}_0$  which causes  $\mathbf{B}_0$  to be a Jordan canonical matrix can be expected to result in a valid solution. Since it is possible for several different choices of  $\mathbf{P}_0$  to satisfy this condition, conceivably there may exist several possible mathematical solutions to the problem.

Since the solution for a step-index fiber exists as a special case of the solution for a graded-index fiber it is reasonable to choose  $\mathbf{P}_0$  based on the knowledge of the exact solution

for a step-index fiber. From the wave equation formulation we know that for a step-index fiber the differential equations for  $E_z$  and  $h_z$  become uncoupled and the resulting equations can be solved independently of each other. This suggests that for the case of a step-index fiber  $\mathbf{P}(s)$  and hence  $\mathbf{P}_0$  should have a form such that two of the four elements in the solution of the vector  $\mathbf{v}(s)$  should contribute to  $E_z$  but not  $h_z$  while the remaining two elements contribute to  $h_z$  only. If  $\mathbf{P}_0$  is chosen as

$$\mathbf{P}_0 = \begin{pmatrix} k_{N1}^2 & 0 & k_{N1}^2 & 0 \\ m\kappa & jm & m\kappa & -jm \\ 0 & k_{N1}^2 & 0 & k_{N1}^2 \\ -jm\epsilon_1 & m\kappa & jm\epsilon_1 & m\kappa \end{pmatrix} \quad (37)$$

then for a step-index fiber  $E_z$  and  $h_z$  are at least uncoupled for the lowest order solution where  $\mathbf{P}(s) = \mathbf{P}_0$ .

Using  $\mathbf{P}_0$  given by eq. (37)  $\mathbf{B}_0$  is given by

$$\mathbf{B}_0 = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix} \quad (38)$$

Since  $\mathbf{B}_0$  is a diagonal matrix, instead of a Jordan canonical matrix as was the case for the transverse modes, eq. (28) can be used to find  $\mathbf{W}_n$ . Recall that this solution for  $\mathbf{W}_n$  may cause some elements of  $\mathbf{W}_n$  to be undefined. In particular, for this problem the elements in the third and fourth columns of both  $\mathbf{W}_{2k}$  and  $\mathbf{P}_{2k}$  are undefined when  $m = 1, 2, \dots, k$ . However, due to the structure of the various matrices and the order of multiplication in the definitions of  $\mathbf{W}_n$  and  $\mathbf{F}_n$  these undefined elements remain in the third and fourth columns of all resulting matrices. In the final solution these undefined elements can be dropped since they contribute only to the two solutions which are not finite at  $s = 0$ .

With  $\mathbf{P}_0$  and  $\mathbf{B}_0$  given by eqs. (37) and (38) respectively, the general form of the solution



to eq. (11a) which is finite at  $s = 0$  is given by

$$\begin{pmatrix} E_z \\ sE_\phi \\ h_z \\ sh_\phi \end{pmatrix} = s^m \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \\ P_{31}(s) & P_{32}(s) \\ P_{41}(s) & P_{42}(s) \end{pmatrix} \begin{pmatrix} C_1 e^{\lambda_1(s)} \\ C_2 e^{\lambda_2(s)} \end{pmatrix} \quad (39a)$$

where

$$P_{ij}(s) = \sum_{n=0}^N (\mathbf{P}_n)_{ij} s^n \quad i = 1, 2, 3, 4; \quad j = 1, 2 \quad (39b)$$

$$\lambda_i(s) = \sum_{n=1}^N (\mathbf{B}_n)_{ii} \frac{s^n}{n} \quad i = 1, 2 \quad (39c)$$

and  $N$  is the number of iterations.

For a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have a parabolic profile given by eq. (30) and  $\epsilon_2(s)$  is a constant, after two iterations the following expressions are found for  $\lambda_i(s)$  and  $\mathbf{P}_{ij}(s)$

$$\lambda_1(s) = -\frac{1}{4m} \left[ \frac{\epsilon_3}{\epsilon_1} (\epsilon_1 - \kappa^2) + \frac{m^2 \kappa^2}{\epsilon_1 - \kappa^2} (2\Delta_1^0) \right] s^2, \quad (40a)$$

$$\lambda_2(s) = -\frac{1}{4m} \left[ (\epsilon_1 - \kappa^2) - \frac{m^2 \epsilon_1}{\epsilon_1 - \kappa^2} (2\Delta_1^0) \right] s^2, \quad (40b)$$

$$P_{11}(s) = (\epsilon_1 - \kappa^2) + \frac{1}{4m(m+1)} \left[ \frac{\epsilon_3}{\epsilon_1} (\epsilon_1 - \kappa^2)^2 - m^2 \kappa^2 (2\Delta_1^0) \right] s^2, \quad (40c)$$

$$P_{12}(s) = -j \frac{m\kappa}{4} \left( \frac{m+2}{m+1} \right) (2\Delta_1^0) s^2, \quad (40d)$$

$$P_{21}(s) = m\kappa + \frac{\kappa}{4(m+1)} \left\{ \frac{\epsilon_3}{\epsilon_1} (\epsilon_1 - \kappa^2) + \frac{m^2 (2\Delta_1^0)}{\epsilon_1 - \kappa^2} \left[ (\epsilon_1 - \kappa^2) + (m+1)\epsilon_1 \right] \right\} s^2, \quad (40e)$$

$$P_{22}(s) = jm - \frac{j}{4(m+1)} \left\{ (\epsilon_1 - \kappa^2) + \frac{m^2 (2\Delta_1^0)}{\epsilon_1 - \kappa^2} \left[ (m+1)\kappa^2 - (\epsilon_1 - \kappa^2) \right] \right\} s^2, \quad (40f)$$

$$P_{31}(s) = -j \frac{m^2 \epsilon_1 \kappa}{4(m+1)} (2\Delta_1^0) s^2, \quad (40g)$$

$$P_{32}(s) = (\epsilon_1 - \kappa^2) + \frac{1}{4m(m+1)} \left[ (\epsilon_1 - \kappa^2)^2 - m^2 \epsilon_1 (2\Delta_1^0) \right] s^2, \quad (40h)$$

$$P_{41}(s) = -jm\epsilon_1 + \frac{j\epsilon_1}{4(m+1)} \left[ \frac{\epsilon_3}{\epsilon_1} (\epsilon_1 - \kappa^2) - \frac{m^2(m+1)\kappa^2}{\epsilon_1 - \kappa^2} (2\Delta_1^0) \right] s^2, \quad (40i)$$

$$P_{42}(s) = m\kappa + \frac{\kappa}{4(m+1)} \left[ (\epsilon_1 - \kappa^2) - \frac{m^2(m+1)\epsilon_1}{\epsilon_1 - \kappa^2} (2\Delta_1^0) \right] s^2. \quad (40j)$$

This should not be considered an accurate solution for  $u(s)$  since the term  $\Delta_3^0$  does not appear anywhere in eqs. (40). This solution is identical to the solution obtained after two iterations for a biaxial graded-index fiber where  $\epsilon_1(s)$  has a parabolic profile and  $\epsilon_2(s)$  and  $\epsilon_3(s)$  are constant. Since  $\Delta_3^0$  only appears in the matrix  $\mathbf{A}_4$  at least four iterations must be performed in order to obtain the effects of a non-constant  $\epsilon_3(s)$ .

The solution for a uniaxial or a step index-fiber can be obtained from eqs. (40) by setting  $\Delta_1^0$  ( and  $\Delta_3^0$  ) equal to zero. Notice that setting  $\Delta_1^0$  equal to zero causes  $P_{12}(s)$  and  $P_{31}(s)$  to be set equal to zero. This corresponds to the decoupling of the differential equations which occurs in the wave equation formulation for the case of a step-index fiber.

From numerical results, it appears that the functions  $\lambda_i(s)$  and  $P_{ij}(s)$  given in eqs. (40) are monotonic. This indicates that the solutions for the various field components will not have an oscillatory behavior. Consequently, for a given value of  $m$  only the mode with the lowest cutoff frequency will be found.

#### IV. Numerical Results

As was previously stated the solution for the biaxial graded-index fiber given by eqs. (40) does not include the effects of a non-constant  $\epsilon_3(s)$ . Obtaining a more accurate solution requires performing more than four iterations. Instead of deriving algebraic equations for the elements of  $\mathbf{F}_n$ ,  $\mathbf{B}_n$ ,  $\mathbf{W}_n$  and  $\mathbf{P}_n$  the values of these matrices can be determined numerically if the values of  $m$ ,  $\kappa$  and  $k_0 a$  are known in advance. One potential difficulty with this method comes from the undefined elements in  $\mathbf{W}_n$  and  $\mathbf{P}_n$ . Since these elements contribute only to the solutions which are unbounded at  $s = 0$  they can be set equal to zero without affecting the final solution. The ability to do this appears to depend upon the form of  $\mathbf{A}(s)$  and the ordering of the eigenvalues of  $\mathbf{A}_0$  in  $\mathbf{B}_0$ .

Asymptotic partitioning was used to solve the matrix equation for several types of fibers. For the case of a step-index fiber a comparison was made between the propagation constants determined using asymptotic partitioning and those determined using the exact solution. For transverse modes the asymptotic solutions were in poor agreement with the exact solutions. Since for transverse modes in a step-index fiber asymptotic partitioning produces a series solution, the poor agreement can be attributed to using too few terms in the series expansion of the exact solution. For hybrid modes there was a much better agreement between the asymptotic solutions and the exact solutions. In particular, for a step-index fiber the asymptotic and the exact solutions produced almost identical values for the propagation constants of the  $\text{HE}_{11}$  mode.

Figure 2, 3 and 4 are plots of the normalized propagation constant for the  $\text{HE}_{11}$ ,  $\text{HE}_{21}$  and  $\text{HE}_{31}$  modes in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have a parabolic

profile and  $\epsilon_2(s)$  is a constant.

## V. Conclusions

For both the wave equation formulation and the matrix equation, exact solutions are known only for the cases of an isotropic step-index fiber and a uniaxial step-index fiber. For isotropic and uniaxial graded-index fibers the wave equation formulation can be solved approximately using WKB analysis. For a biaxial graded-index fiber WKB analysis can not be used on the wave equation formulation. Asymptotic partitioning can be used to solve the matrix equation for all types of permittivity profiles. For meridional modes asymptotic partitioning appears to generate the series solution for the matrix differential equation. For hybrid modes, of the solutions produced by asymptotic partitioning have a form such that for a given value of  $m$  only the mode with the lowest cutoff frequency can be found. A nice feature of the asymptotic solutions is that they remain valid all the way down to cutoff.

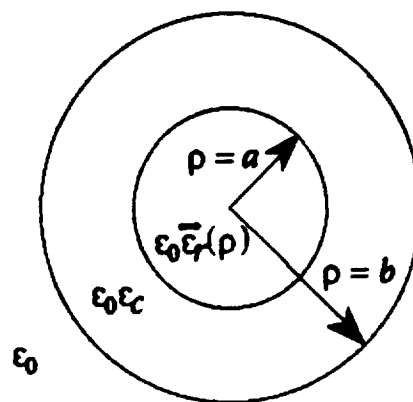
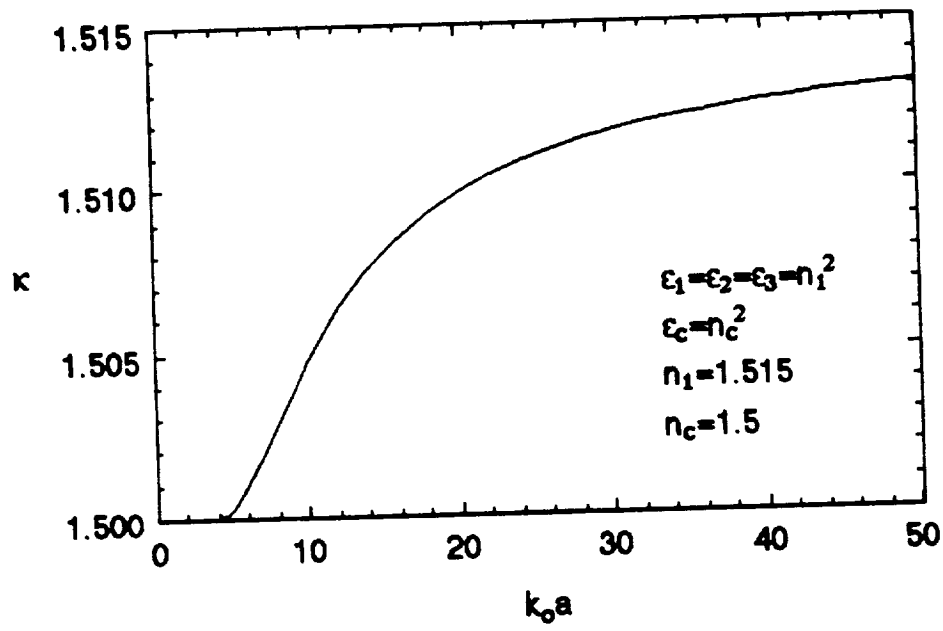
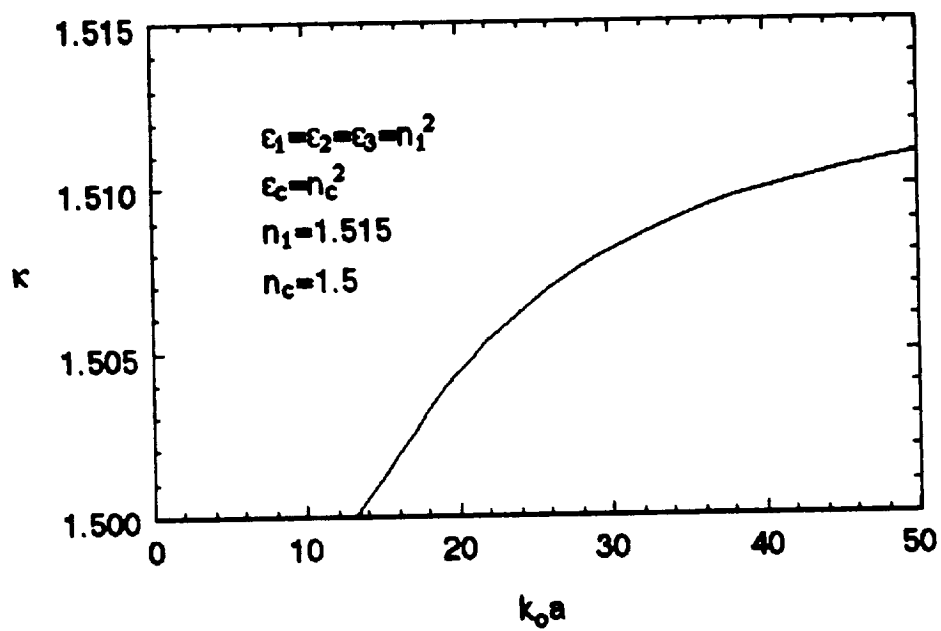


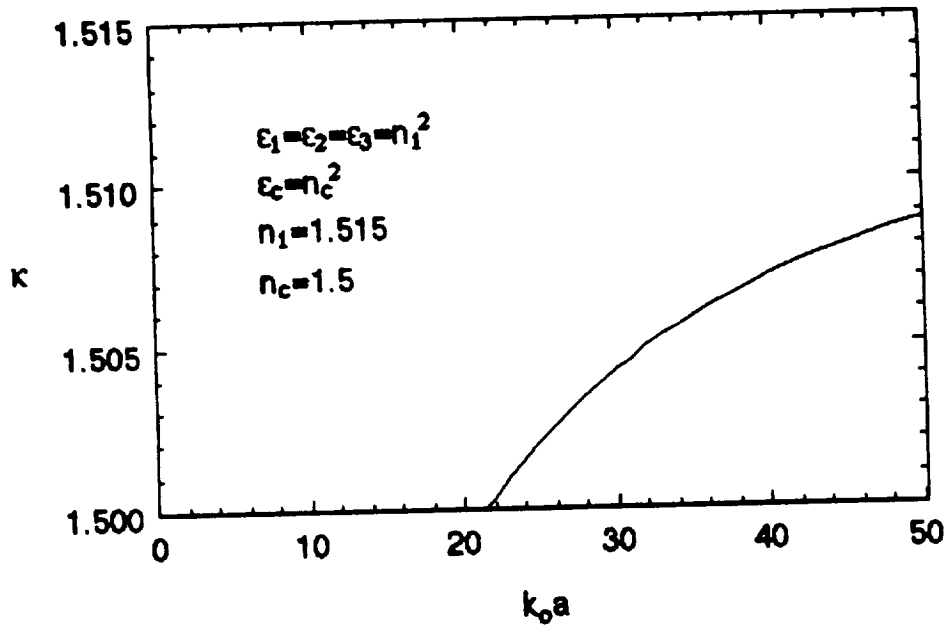
Figure 1 Geometry of the fiber



**Figure 2** Normalized propagation constant for the  $HE_{11}$  mode in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have the parabolic profile of eq. (30) and  $\epsilon_2(s)$  is a constant.



**Figure 3** Normalized propagation constant for the  $HE_{21}$  mode in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have the parabolic profile of eq. (30) and  $\epsilon_2(s)$  is a constant.



**Figure 4** Normalized propagation constant for the  $HE_{31}$  mode in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have the parabolic profile of eq. (30) and  $\epsilon_2(s)$  is a constant.



## References

1. E. Snitzer, "Cylindrical dielectric waveguide modes," *J. Opt.Soc. Amer.*, vol. 51, pp. 491-498, 1961
2. D.K. Paul and R.K. Shevgaonkar, "Multimode propagation in anisotropic optical waveguides," *Radio Sci.*, vol. 16, pp. 525-533, 1981.
3. P.K. Tien, J.P. Gordon, J.R. and Whinnery, "Focusing of a light beam of gaussian field distribution in continuous and parabolic lens-like media," *Proc. IEEE*, vol. 53, pp. 129-136, 1965
4. W. Streifer C.N. Kurtz, "Scalar analysis of radially inhomogeneous guiding media," *J. Opt. Soc. Am.*, vol. 57, pp. 779-789, 1967
5. A.H. Cherin, *An Introduction to Optical Fibers*. New York: McGraw Hill, 1983
6. A. Yariv, *Optical Electronics*. New York: Holt, Reinhart and Winston, Inc., 1985
7. C. Kurtz and W. Streifer, "Guided waves in inhomogeneous focusing media Part I: Formulation, solution for quadratic inhomogeneity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 11-15, 1969
8. C. Kurtz and W. Streifer, "Guided Waves in inhomogeneous focusing media Part II: Asymptotic solution for general weak inhomogeneity," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 250-253, 1969
9. G.L. Yip and S. Nemoto, "The relations between scalar modes in a lenslike medium and vector modes in a self-focusing optical fiber," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 260-263, 1975
10. M. Hashimoto, "Asymptotic vector modes in inhomogeneous circular waveguides," *Ra-*

- dio Sci.*, vol. 17, pp. 3-9, 1982
11. H. Ikuno, "Vectorial wave analysis of graded-index fibers," *Radio Sci.*, vol. 17, pp. 37-42, 1982
  12. A. Tønning, "Circularly symmetric optical waveguide with strong anisotropy," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 790-794, 1982
  13. C. Yeh and G. Lindgren, "Computing the propagation characteristic of radially stratified fibers: an efficient method," *Applied Optics*, vol. 16, pp. 483-493, 1977
  14. A. Tønning, "An alternative theory of optical waveguides with radial inhomogeneities," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-30, pp. 781-789, 1982
  15. J. Mathews and R.L. Walker, *Mathematical Methods of Physics*. New York: Benjamin/Cummings, 1964
  16. A. Nayfeh, *Perturbation Methods*. New York: John Wiley & Sons, Inc., 1973

## Figure Captions

**Figure 1** Geometry of the fiber

**Figure 2** Normalized propagation constant for the  $HE_{11}$  mode in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have the parabolic profile of eq. (30) and  $\epsilon_2(s)$  is a constant.

**Figure 3** Normalized propagation constant for the  $HE_{21}$  mode in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have the parabolic profile of eq. (30) and  $\epsilon_2(s)$  is a constant.

**Figure 4** Normalized propagation constant for the  $HE_{31}$  mode in a biaxial graded-index fiber where  $\epsilon_1(s)$  and  $\epsilon_3(s)$  have the parabolic profile of eq. (30) and  $\epsilon_2(s)$  is a constant.